Journal of Engineering Physics and Thermophysics, Vol. 82, No. 2, 2009

THE VELOCITY OF MOVEMENT OF ISOTHERMS NEAR THE THERMAL CENTER OF A COOLED BODY

V. V. Vinogradov,[a] **I. L. Tyazhel'nikova,**[a] **and E. P. Vinogradova**[b]

UDC 536.2

The behavior of isotherms near the thermal center of a cooled body has been considered. It is shown that closer to the thermal center the velocity of movement of isotherms increases sharply. An expression is obtained that describes the behavior of the velocity of movement of isotherms near the center of bodies of simple shapes (plate, infinite cylinder, rod of square cross section, sphere, and cube). Quantitatively the behavior of the velocity depends on the thermal diffusivity of the medium and the geometric form of the body but is independent of the absolute magnitude of the isotherms, body dimensions, and of the cooling inetnsity. The property indicated is typical of the processes described by parabolic-type equations.

Keywords: *velocity of movement of isotherms, thermal center of a cooled body, plate, cylinder, sphere.*

Introduction. The velocity of movement of isotherms during thermal treatment not only reflects the kinetics of cooling (heating) of metal but in a number of cases influences the characteristic features of the structure being formed. Near a metal surface the velocity of movement of isotherms is determined by the conditions of cooling of its surface and correspondingly relatively simply lends itself to control (regulation). However, the behavior of the isotherms in the inside regions is more conservative because of their specific features. In this paper we study this behavior by analytical and numerical methods.

Statement of the Problem (Plate). We begin the study by an analysis of the process of cooling of an infinite plate of thickness $2L$ (one-dimensional problem); the problem is symmetrical, the center of the system of Cartesian coordinates is at the middle of the plate, and the x axis is perpendicular to the plate surface. The heat conduction equation, initial and boundary conditions have the form

$$\frac{\partial T(x,t)}{\partial t} = a\frac{\partial^2 T(x,t)}{\partial x^2}\,, \tag{1}$$

$$T(x, t=0) = T_0\,, \tag{2}$$

$$T(x=\pm L, t) = T_s\,. \tag{3}$$

Equaiton (1) subject to conditions (2) and (3) was solved numerically on a PC using the standard method of finite differences [1]. The thermal diffusivity, initial and boundary conditions, and the values of the three analyzed isotherms ($T_1 > T_2 > T_3$) were given for the steel plate: a = 0.05 cm^2/sec, L = 10 cm, T_0 = 1050^{o}C, T_s = 500^{o}C, T_1 = 1000^{o}C, T_2 = 800^{o}C, and T_3 = 600^{o}C.

Discussion of the Results of Numerical Solution (Plate). Figure 1 presents the results of calculation of the velocity of movement of the indicated isotherms depending on the depth of their penetration into the plate. With allowance for the symmetry of the problem, the results are presented for a half-plate. Near the plate surface (x = L) the

[a]I. P. Bardin Central Scientific-Research Institute of Ferrous Metallurgy, 9/23 2nd Bauman Str., Moscow, 105005, Russia; [b]V. A. Trapeznikov Institute of the Problems of Management, Russian Academy of Sciences, 65 Profsoyuznaya Str., Moscow, 117806, Russia; email: vvvinogradov@mtu-net.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 82, No. 2, pp. 391–397, March–April, 2009. Original article submitted January 29, 2008; revision submitted June 19, 2008.

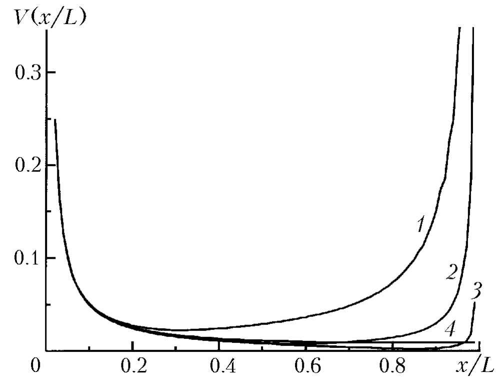

Fig. 1. Velocity of moment of isotherms $T = T_1$ (curve 1), $T = T_2$ (curve 2), $T = T_3$ (curve 3), and $T = T_1$ in the case of slow cooling (curve 4). V, cm/sec.

behavior of the isotherms is determined by the eigenvalue of an isotherm and by the condition of plate cooling prescribed by the boundary condition. In our case of boundary condition (3) the velocities of movement of isotherms near the surface have the highest values, which rapidly diminish with distance from the surface. Thus, the high velocities of the movement of isotherms near the plate surface are related to the jumpwise change in the plate surface temperature from the initial condition (2) to condition (3). In the case of a monotonic change in the temperature on the plate surface, for example, when a linear law of the form of $T(x = \pm L, t) = T_0 - vt$ $(v > 0)$ holds, this feature virtually disappears. This is illustrated in Fig. 1 by curve 4 obtained for the $T = T_1$ isotherm at $v = 0.1^{\circ}$ C/sec. With increase in the distance from the surface the velocity of movement of isotherms decreases, and this occurs the earlier and more intensely the lower the value of the isotherm. After the decrease, the velocity of movement of isotherms begins to rise — the earlier the lower the isotherm. At a great distance from the plate surface $(x \le 0.3L)$ the velocity of movement of all isotherms increases sharply — the more rapidly the closer the isotherm to the plate middle. At the center of the plate (~20% of its width) the velocity of movement of all isotherms becomes practically identical. In the axial zone of the plate the behavior of isotherms no longer depends on the eigenvalue of an isotherm and cooling intensity, as vividly illustrated by the behavior of curves 1–4 in Fig. 1. The reason for such an acceleration in the movement of isotherms closer to the plate center becomes clear if one takes into account that the temperature field of the cooled plate is described by a dome-shaped curve with a flat portion at the plate center [2, 3]. In this case, when an isotherm approaches the center of the plate, the heat flux from the central parts decreases and tends to zero, whereas the outward flux increases more and more rapidly, thus leading to the ever-increasing velocity of movement of isotherms.

Asymptotic Solution (Plate). In order to analyze the asymptotic behavior of the velocity of movement of isotherms when they approach the plate center (with $x \to 0$), and to determine the functional dependence of the velocity on the process parameters, we will avail ourselves of the classical, e.g. [2, 3], analytical solution of problem (1)–(3). According to [2, 3], the temperature distribution in the plate at each moment is described by the expression

$$T(x, t) = T_s + \frac{4(T_0 - T_s)}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left(-\frac{a(2n+1)^2 \pi^2 t}{4L^2}\right) \cos\left(\frac{(2n+1)\pi x}{2L}\right). \tag{4}$$

From numerical calculations and relation (4) it follows that for isotherms at the middle of the plate $\mathrm{Fo} = at/x^2 \ge 0.3$; following [3], this allows one to restrict oneself in Eq. (4) to the first term of the series (regular regime). Then, retaining the first three terms in the expansion of the cosine, we obtain a simple biquadratic equation for the coordinate x:

$$\tilde{x}^4 - 12\tilde{x}^2 + 24(1 - \tilde{T} \exp \tilde{t}) = 0\,, \tag{5}$$

where $\tilde{x} = \frac{x}{L}\frac{\pi}{2}$; $\tilde{T} = \frac{4}{\pi}\frac{T - T_s}{T_0 - T_s}$; $\tilde{t} = \frac{\pi^2 at}{4L^2}$.

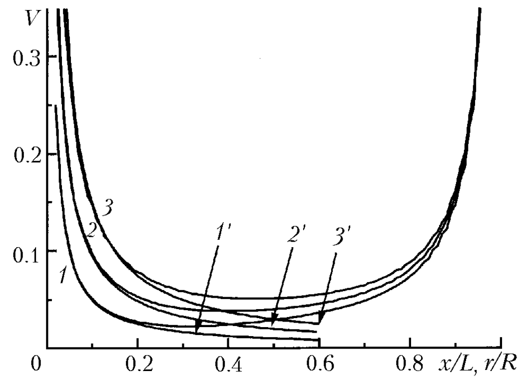

Fig. 2. Velocity of movement of the isotherm $T = T_1$ in a plate — curve 1, in a cylinder — 2, in a sphere — 3; 1′, 2′, and 3′) asymptotic solutions near the thermal center of the corresponding bodies. V, cm/sec.

Solving the biquadratic equation (5) and discarding the solutions that do not satisfy the physical conditions, for the dimensionless coordinate $\tilde{x}$ we find the expression that describes the movement of isotherms near the middle of the plate at each instant of time $\tilde{t}$:

$$\tilde{x} = \sqrt{6}\left(1 - \sqrt{\frac{1}{3}(1 + 2\tilde{T}\exp\tilde{t})}\right)^{1/2}. \tag{6}$$

Differentiating (6) with respect to time $\tilde{t}$, we obtain the expression for the velocity of movement of isotherms when they approach the center of the plate:

$$\dot{\tilde{x}} = -\frac{\sqrt{2}}{2}\tilde{T}\exp\tilde{t}\left[(1 + 2\tilde{T}\exp\tilde{t})\left(1 - \sqrt{\frac{1}{3}(1 + 2\tilde{T}\exp\tilde{t})}\right)\right]^{-1/2}. \tag{7}$$

Passing to dimensional variables, with allowance for Eq. (6) and the fact that in the middle of the plate $\tilde{T}$ exp $\tilde{t} \to 1$, we have

$$\dot{x} = -K_{\text{sh}}\frac{a}{x}, \tag{8}$$

where the derivative is taken over the real time t, the minus sign denotes that, with increase in time, the coordinate x decreases. For the plate $K_{\text{sh}} = 1$, and further it will be shown that its value depends on the shape of the body cooled, and an explanation for bodies of simple shapes will be given.

Discussion of Results (Plate). For comparative purposes Fig. 2 presents the dependence of the velocity of movement of the isotherm $T = T_1$ vs. the depth of penetration into the plate x obtained by numerical solution of Eq. (1) with conditions (2) and (3) (curve 1) and asymptotic solution (8) (curve 1′). It is seen from this figure that the analytical solution (8) absolutely exactly coincides with the numerical solution of the problem at the final stage. Formula (8) is the asymptotic expression (when $x \to 0$) for the velocity of movement of isotherms. It follows from this expression that, when the isotherms approach the center of the plate, their velocity may acquire infinitely high values. This does not contradict the physical laws, since the velocities of movement are related not to the displacement of a physical substance, but are realized because of the local transformations of the temperature field of the plate. In connection with this, the noted features in the behavior of the isotherms cannot be related to paradoxes (incorrectnesses) of the classical heat conduction equaiton that are discussed in [4].

We will note some of the trends typical of this phenomenon and following from the numerical solution of the equaiton of type (1) subject to conditions (2), (3) and of analytical expression (8). First of all we note that with a change in the intensity of cooling of the plate the velocity of movement of isotherms near the middle of the plate does

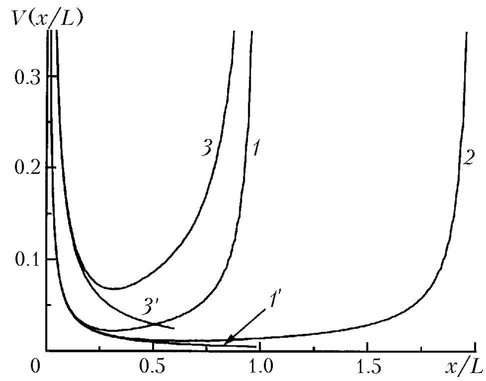

Fig. 3. Velocity of movement of the isotherm $T = T_1$: in a plate with $a = 0.05$ cm^2/sec of thickness $L = 10$ cm (curve 1) and of thickness $2L$ (curve 2); 1′) the asymptotics general for them; in a plate with $a = 0.15$ cm^2/sec with $L =$ 10 cm (curve 3); 3′) corresponding asymptotic solution. V, cm/sec.

not change, which is confirmed by the behavior of curves 1 and 4 in Fig. 1. Also independent of the plate thickness is the velocity of movement of isotherms near the thermal center. This is proved, for example, by the complete coincidence of the behavior of the isotherm $T = T_1$ in the axial zone of the plate of thickness $L = 10$ cm (Fig. 3, curve 1) and in the plate of thickness $2L$ (Fig. 3, curve 2); curve 1′ (Fig. 3) is their common asymptotic solution (8). However, in the material with a higher thermal diffusivity the velocity of movement of isotherms at the concluding stage is higher; it increases proportionally to the increase in the thermal diffusivity. This is illustrated by curves 1 and 3 in Fig. 3 obtained for the isotherm $T = T_1$: curve 1 was obtained at $a = 0.05$ cm^2/sec (1′ is the asymptotic solution) and curve 3 at $a = 0.15$ cm^2/sec (3′ is the asymptotic solution obtained from Eq. (8) for the variant with $a = 0.15$ cm^2/sec). The proportionality is confirmed by the results of numerical solution and by the behavior of the asymptotic curves. Calculations have also shown that the velocity of movement of isotherms at the center of the plate depends on the geometric shape of a specimen. We will consider this fact in more detail.

Statement of the Problem (Cylinder). We will consider the cooling of an infinite circular cylinder of radius R. The cylindrical coordinate system is fixed on the cylinder axis. The heat conduction equation and the initial and boundary conditions have the form [2, 3]

$$\frac{\partial T(r,t)}{\partial t} = \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,t)}{\partial r}\right), \tag{9}$$

$$T(r, t = 0) = T_0\,, \tag{10}$$

$$T(r = R, t) = T_s\,, \tag{11}$$

$$T(r = 0, t) \neq \infty\,. \tag{12}$$

By analogy with the previous case, Eq. (9), subject to conditions (10)–(12), was solved numerically and analytically. The numerical solution obtained for the isotherm $T = T_1$ at $a = 0.05$ cm^2/sec and at the same previous values of T_0 and T_s in the boundary conditions (10) and (11) are given in Fig. 2 (curve 2). From this figure it is seen that near the surface the solutions for the cylinder and the plate coincide.

Asymptotic Solution (Cylinder). For a more detailed analysis of the behavior of the velocity of movement of isotherms when they approach the center of the infinite cylinder, and for determining the functional dependence of the velocity on the parameters of the process, we will find the asymptotic solution (when $r \to 0$) for the velocity. For this purpose, we will use the well-known analytical solution of problem (9)–(12). According to [2, 3, 5, 6], the distribution of temperature in an infinite cylinder at each moment t is described by the expression

$$\frac{T(r,t)-T_s}{T_0-T_s}=\sum_{n=1}^{\infty}\frac{2}{\mu_n J_1(\mu_n)}\exp\left(-\frac{\mu_n^2 at}{R^2}\right)J_0\left(\mu_n\frac{r}{R}\right), \tag{13}$$

where μ_n are the roots of the characteristic equaiton $J_0(\mu) = 0$: $\mu_1 = 2.4048$, $\mu_2 = 5.5101$, $\mu_3 = 8.6537$ and so on; $J_1(\mu_1) = 0.5202$ [2, 3, 7]. As is known [3, 5], series (13), except for the small Fourier numbers, is rapidly divergent, and therefore we will restrict ourselves to its first term, whence we have

$$\frac{T(r,t)-T_s}{T_0-T_s}=\frac{2}{\mu_1 J_1(\mu_n)}\exp\left(-\frac{\mu_1^2 at}{R^2}\right)J_0\left(\mu_1\frac{r}{R}\right). \tag{14}$$

For the region of small values of r/R we will use the well-known [8] expansion of the Bessel functions into a series:

$$J_\nu(z)=\left(\frac{z}{2}\right)^{\nu}\sum_{n=1}^{\infty}\frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(\nu+n+1)}, \tag{15}$$

where for the natural numbers $\Gamma(1) = 1$, $\Gamma(2) = 1$, $\Gamma(n) = (n-1)!$. Taking into account, the first three terms in expansion (15), from Eq. (14) we obtain

$$\tilde{T}=1.6\exp(-\tilde{t})\left[1-\left(\frac{\tilde{r}}{2}\right)^2+\frac{1}{2}\left(\frac{\tilde{r}}{2}\right)^4\right], \tag{16}$$

where $\tilde{T} = \frac{T(r,t)-T_s}{T_0-T_s}$; $\tilde{r} = \mu_1\frac{r}{R}$; $\tilde{t} = \frac{\mu_1^2 at}{R^2}$.

Solving Eq. (16) for $\tilde{r}$, we have

$$\tilde{r}^4-8\tilde{r}^2+32\left(1-1.6\tilde{T}\exp\tilde{t}\right)=0\,. \tag{17}$$

Discarding the solutions of Eq. (17) that do not have physical meaning, we obtain

$$\tilde{r}=2\left(1-\sqrt{3.2\,\tilde{T}\exp(\tilde{t})-1}\right)^{1/2}. \tag{18}$$

Differentiation of Eq. (18) with respect to the dimensionless time $\tilde{t}$ yields

$$\tilde{r}=-1.6\,\tilde{T}\exp\tilde{t}\left(\left(3.2\,\tilde{T}\exp\tilde{t}-1\right)\left(1-\sqrt{3.2\,\tilde{T}\exp\tilde{t}-1}\right)\right)^{-1/2}. \tag{19}$$

Passing to dimensional variables with account for Eq. (18) and for the fact that with approach to the cylinder axis $\tilde{T}\exp\tilde{t}\to 0.625$, we have

$$\dot{r}=-K_{sh}\frac{a}{x}, \tag{20}$$

where the coefficient of the shape $K_{sh} = 2$.

Discussion of General Results. The obtained asymptotic solution (20) is given in Fig. 2 (curve 2′), from which it is seen that near the axis of the cylinder it entirely coincides with numerical solution (9)–(12) obtained earlier

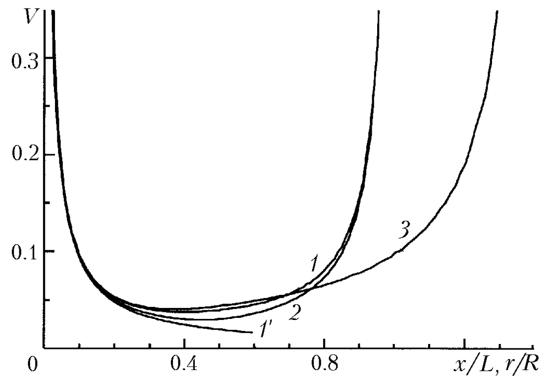

Fig. 4. Velocity of movement of the isotherm $T = T_1$ in the cylinder (curve 1), in a bar of square cross section in the direction perpendicular to the surface (curve 2) and along the diagonal of the bar cross section (curve 3); 1′) corresponding general asymptotic solution. V, cm/sec.

for the isotherm $T = T_1$ (curve 2). Analogously a numerical solution was obtained, as well as an asymptotic one that describes an accelerated movement of isotherms in the axial zone in the form identical with that of (8), (20) during cooling of a body of spherical shape (sphere) and of a cube (for them $K_{sh} = 3$) and of an infinite bar of quadratic cross section (for which $K_{sh} = 2$). For illustration Fig. 2 (curve 3) demonstrates the behavior of the velocity of movement of the isotherm $T = T_1$ in a cooled sphere of radius $R = 10$ cm ($a = 0.05$ cm^2/sec, obtained by numerical solution of the heat conduction equaiton for a sphere [2, 3] with corresponding boundary-value conditions of type (10)–(12) and asymptotic solution (20) with $K_{sh} = 3$ (curve 3′) that entirely coincide near the sphere center. Figure 4 presents also the results of the numerical solution of the classical heat conduction equaiton [2, 3] with the corresponding boundary-value conditions of type (10)–(12) for a cooled infinite bar of quadratic cross section and thickness $L =$ 10 cm with $a = 0.05$ cm^2/sec. Shown here is the behavior of the velocity of movement of the isotherm ($T = T_1$) in the section of the bar perpendicular to its surface along the shortest direction (curve 2) and along the diagonal of the section (curve 3). For comparison the figure presents the velocity of movement of the isotherm $T = T_1$ in an infinite circular cylinder with $R = 10$ cm (curve 1) and an asymptotic solution (20) with $K_{sh} = 2$ (curve 1′). It is clearly seen from the figure that in the central zone the velocities of movement of the isotherm ($T = T_1$) in a circular cylinder and in a square bar (along the normal and the diagonal) entirely coincide with each other and with the asymptotic solution (20) at $K_{sh} = 2$. Analogous results were also obtained for a cooled cube for which the asymptotic solution also has the form of (20), but the proportionality factor is $K_{sh} = 3$.

If we take into consideration that the intensity of the processes proceding in cooled bodies [3, 6] is proportional to the magnitude of the cooling surface S_b and inversely proportional to the volume V_b, it is possible to formulate the following principle of calculation of the coefficient K_{sh} for bodies of simple shape in the asymptotic expression (20):

$$K_{sh} = \frac{S_b}{V_b}\frac{A}{2}, \tag{21}$$

where A is the characteristic dimension of the body (the thickness of the plate and of the bar of square cross section, the diameter of the cylinder and sphere, the side of the cube).

Integrating expressions of the type of (8) and (20) and taking into account that at $t = t_f$ $x = 0$ (t_f is the time when an isotherm reaches the symmetry center of the body), we have

$$x = \sqrt{2aK_{sh}(t_f - t)}, \tag{22}$$

where $K_{sh} = 1$ for a plate, 2 for a circular cylinder and a bar of square cross section, and 3 for a body of spherical shape and cube. From this it follows that close to the symmetry center of the body the movement of isotherms obeys the law of the root (22) which is a simplified analog of expressions (6) and (18).

On the whole, an analogous picture in the acceleration of the movement of isotherms closer to the symmetry center of a body is observed if the body is heated instead of cooled. This analysis can also be related to the velocities of movement of isotherms that correspond to solid-body transformations proceding during cooling (heating) of steels without a noticeably expressed thermal effect, as, e.g., is the case in thermal treatment of steels. Moreover, since the diffusional processes, just as the thermal ones, are described by parabolic-type equations, the results obtained can be related also to the processes of diffusional type proceding in a solid-body, e.g., to the velocity of moment of isoconcentrates.

CONCLUSIONS

1. In cooled bodies of a simple shape (plate, circular cylinder, bar of square cross section, sphere, and cube) the movement of isotherms toward the symmetry center of a sample (thermal center) has a clearly expressed accelerated character which is a universal property of the processes, the progress of which is described by parabolic-type partial differential equations.

2. Quantitatively the velocity of moment of isotherms in such bodies near the symmetry center is determined by the thermal diffusivity of the medium and by the geometric shape of the sample and is independent of the body size and intensity of cooling. A simple analytical expression for the velocity of movement of isotherms near the thermal center of such bodies has been obtained.

NOTATION

A, characteristic dimension, cm; a, specific thermal diffusivity of the material, cm^2/sec; Fo, Fourier number; J_0, J_n, J_ν, Bessel functions of the zero, nth order and of index ν; K_{sh}, dimensionless coefficient of shape; L, plate thickness, cm; n, natural number; R, radius of a sample, cm; r, radial coordinate, cm; S_b, magnitude of the cooled surface, cm^2; T, temperature, ^{o}C; T_0, T_s, T_i, initial temperature, temperature of the body surface, and characteristic temperatures (i = 1, 2, 3), ^{o}C; t, time, sec; t_f, time when the isotherm reaches the symmetry center, sec; v, rate of change in the body surface temperature, ^{o}C/sec; $\dot{x}$, $\dot{r}$, velocity of movement of isotherms, cm/sec; V_b, volume of the body, cm^3; x, coordinate, cm; z, argument of the Bessel function; Γ, gamma-function; μ_i, roots of the characteristic equation $J_0(\mu) = 0$. Subscripts and superscripts: b, body, f, finite moment; s, wall (surface); sh, shape (form); 0, initial moment; $\tilde{}$, nondimensionality of a variable.

REFERENCES

1. A. A. Samarskii, *Theory of Difference Schemes* [in Russian], Nauka, Moscow (1977).
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* [Russian translation], Nauka, Moscow (1964).
3. Luikov, *Heat Conduction Theory* [in Russian], Vysshaya Shkola, Moscow (1967).
4. A. G. Shashkov, V. A. Bubnov, and S. Yu. Yanovskii, *Wave Phenomena of Heat Conduction* [in Russian], Editorial, Moscow (2006).
5. A. D. Polyanin, A. V. Vyaz'min, A. I. Zhurov, and D. A. Kazenin, *Handbook on Exact Solutions of Heat- and Mass Transfer Equations* [in Russian], Faktorial, Moscow (1998).
6. V. P. Isachenko, V. A. Osipova, and A. S. Sukomel, *Heat Transfer* [in Russian], Énergiya, Moscow (1975).
7. E. Janke, F. Emde, and F. Lösch, *Tafel noherer Funktionen* [Russian translation], Nauka, Moscow (1977).
8. B. G. Korenev, *Introduction to the Theory of Bessel Functions* [in Russian], Nauka, Moscow (1971).